BIBLIOTECA DE MATEMATICĂ

Redactare: Ramona Rossall, Iuliana Ene
Tehnoredactare: Mioara Benza
Pregătire de tipar: Marius Badea
Design copertă: Marius Badea

Descrierea CIP a Bibliotecii Naţionale a României
CHIRCIU, MARIN
Inegalităţi geometrice - 2 : de la iniţiere la performanţă /
Marin Chirciu. - Piteşti : Paralela 45, 2021
Conţine bibliografie
ISBN 978-973-47-3210-4

514

COMENZI – CARTEA PRIN POŞTĂ

EDITURA PARALELA 45
Bulevardul Republicii, Nr. 148, Clădirea C1, etaj 4, Pitești,
jud. Argeș, cod 110177
Tel.: 0248 633 130; 0753 040 444; 0721 247 918
Tel./fax: 0248 214 533; 0248 631 439; 0248 631 492
E-mail: **comenzi@edituraparalela45.ro**
sau accesaţi **www.edituraparalela45.ro**

Tiparul executat la tipografia *Editurii Paralela 45*
E-mail: tipografie@edituraparalela45.ro

Marin Chirciu

INEGALITĂŢI GEOMETRICE 2

DE LA INIŢIERE LA PERFORMANŢĂ

Editura Paralela 45

Cuprins

capitolul 1

Metoda SOS

„Când apare un om deştept pe lume, poţi să-l recunoşti după faptul că toţi proştii îl urăsc.”
Bisanne de Soleil

Sunt adevărate afirmaţiile:

$a^2 \geq 0,\ \forall a \in \mathbb{R}$, cu egalitate pentru $a = 0$.

$(a-b)^2 \geq 0,\ \forall a,\ b \in \mathbb{R}$, cu egalitate pentru $a = b$.

$a^2 + b^2 + c^2 \geq ab + bc + ca,\ \forall a,\ b,\ c \in \mathbb{R}$, cu egalitate pentru $a = b = c$.

$(a+b+c)^2 \geq 3(ab+bc+ca),\ \forall a,\ b,\ c \in \mathbb{R}$, cu egalitate pentru $a = b = c$.

$ax^2 + bx + c \geq 0,\ \forall x \in \mathbb{R} \Leftrightarrow \begin{cases} a > 0 \\ \Delta \leq 0 \end{cases}$, cu egalitate pentru $\Delta = 0$ şi $x = \dfrac{-b}{2a}$.

$a \leq x \leq b \Leftrightarrow (x-a)(x-b) \leq 0$, cu egalitate dacă $x = a$ sau $x = b$.

$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca),\ \forall a,\ b,\ c \in \mathbb{R}$.

$a^3 + b^3 + c^3 \geq 3abc,\ \forall a,\ b,\ c \geq 0$, cu egalitate pentru $a = b = c$.

$a^2 + b^2 + c^2 \geq a + b + c,\ \forall a,\ b,\ c > 0$ şi $abc = 1$, cu egalitate pentru $a = b = c = 1$.

$(a+b+c)\left(\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\right) \geq 9,\ \forall a,\ b,\ c > 0$, cu egalitate pentru $a = b = c$.

În continuare sunt propuse aplicaţii ce pot fi rezolvate folosind inegalităţile de mai sus.

Să se arate că în orice triunghi *ABC* sunt adevărate inegalităţile:

1.1. a) $\sum \left| 3a^2 - 2b^2 \right| \cdot m_a \geq \dfrac{8S^2}{R}$.

Mathematical Reflections, Titu Andreescu, Dallas, SUA

b) $\sum \left| 2a^2 - b^2 \right| \cdot m_a \geq \dfrac{8S^2}{R}$.

c) $\sum\left|\,(n+1)a^2-nb^2\,\right|\cdot m_a \ge 4(n+2)\cdot\frac{S^2}{R}\ge\frac{8S^2}{R}$, unde $0\le n\le 2$.

d) $\sum\left|\,xa^2-yb^2\,\right|\cdot m_a \ge \left|\,x-y\,\right|\cdot\frac{8S^2}{R}$, unde $2x\ge 3y\ge 0$.

e) $\sum\left|\,3a^2-2b^2\right|\cdot(m_b+m_c)\ge\frac{16S^2}{R}$.

f) $\sum\left|\,2a^2-b^2\right|\cdot(m_b+m_c)\ge\frac{16S^2}{R}$.

g) $\sum\left|\,(n+1)a^2-nb^2\right|\cdot(m_b+m_c)\ge\frac{16S^2}{R}$, unde $-2\le n\le 2$.

h) $\sum\left|\,xa^2-yb^2\right|\cdot(m_b+m_c)\ge\left|\,x-y\,\right|\cdot\frac{16S^2}{R}$, unde $2x\ge 3y\ge 0$.

i) $\sum\left|\,3a^2-2b^2\right|\cdot(m_b+\lambda m_c)\ge(1+\lambda)\frac{8S^2}{R}$, unde $\lambda\ge 1$.

j) $\sum\left|\,2a^2-b^2\right|\cdot(m_b+\lambda m_c)\ge(1+\lambda)\frac{8S^2}{R}$, unde $\lambda\ge 0$.

k) $\sum\left|\,(n+1)a^2-nb^2\right|\cdot(m_b+\lambda m_c)\ge(1+\lambda)\frac{8S^2}{R}$, unde $-2\le n\le 2$ şi $\lambda\ge n-1$.

l) $\sum\left|\,xa^2-yb^2\right|\cdot(m_b+\lambda m_c)\ge\left|\,x-y\,\right|\cdot(1+n)\frac{8S^2}{R}$, unde $(1+\lambda)x\ge(2+\lambda)y\ge 0$ şi $\lambda\ge 0$.

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1.2. $\frac{1}{(\cos A+\cos B)^2}+\frac{1}{(\cos B+\cos C)^2}+\frac{1}{(\cos C+\cos A)^2}\ge 3$.

D.M. Bătineţu-Giurgiu şi Neculai Stanciu

1.3. a) $\frac{m_a+m_b+m_c}{\sin^2 A+\sin^2 B+\sin^2 C}\ge 4r$.

George Apostolopoulos, Messolonghi, Grecia

b) $\frac{m_a+m_b+m_c}{\sin^2 A+\sin^2 B+\sin^2 C}\ge 2R\ge 4r$.

1.4. a) $\frac{a(b+c)}{bc\cdot\cos^2\frac{A}{2}}+\frac{b(c+a)}{ca\cdot\cos^2\frac{B}{2}}+\frac{c(a+b)}{ab\cdot\cos^2\frac{C}{2}}\ge 8$.

Daniel Sitaru

b) $\frac{a(b+c)}{bc\cdot\sin^2\frac{A}{2}}+\frac{b(c+a)}{ca\cdot\sin^2\frac{B}{2}}+\frac{c(a+b)}{ab\cdot\sin^2\frac{C}{2}}\ge 12$.

c) $\dfrac{a(b+c)}{bc\cdot\sin^2 A}+\dfrac{b(c+a)}{ca\cdot\sin^2 B}+\dfrac{c(a+b)}{ab\cdot\sin^2 C}\ge 8$.

d) $\dfrac{a(b+c)}{bc\cdot\sin A}+\dfrac{b(c+a)}{ca\cdot\sin B}+\dfrac{c(a+b)}{ab\cdot\sin C}\ge 4\sqrt{3}$.

e) $\dfrac{a(b+c)}{bc\cdot\sin^3 A}+\dfrac{b(c+a)}{ca\cdot\sin^3 B}+\dfrac{c(a+b)}{ab\cdot\sin^3 C}\ge \dfrac{16}{\sqrt{3}}$.

f) $\dfrac{a(b+c)}{bc\cdot\sin^4 A}+\dfrac{b(c+a)}{ca\cdot\sin^4 B}+\dfrac{c(a+b)}{ab\cdot\sin^4 C}\ge \dfrac{32}{3}$.

Dezvoltări, Marin Chirciu

1.5. a) $\sum\dfrac{b^4+c^4}{\operatorname{tg}^2\dfrac{B}{2}+\operatorname{tg}^2\dfrac{C}{2}}\ge 48S^2$. *George Apostolopoulos, Messolonghi, Grecia*

b) $\sum\dfrac{b^4+nc^4}{\operatorname{tg}^2\dfrac{B}{2}+n\operatorname{tg}^2\dfrac{C}{2}}\ge 48S^2$, unde $n\ge 0$.

1.6. $\dfrac{9R}{8S}-\sum\dfrac{1}{a+b}\ge\sum\dfrac{(a-b)^2}{2(p-a)(p-b)(a+b+2c)}$.

Neculai Stanciu, Buzău şi Titu Zvonaru, Comăneşti

1.7. Arătaţi că în orice triunghi ascuţitunghic au loc următoarele relaţii:

a) $\dfrac{a^2}{\operatorname{tg}B+\operatorname{tg}C}+\dfrac{b^2}{\operatorname{tg}C+\operatorname{tg}A}+\dfrac{c^2}{\operatorname{tg}A+\operatorname{tg}B}\le pR$.

Daniel Sitaru

b) $\dfrac{a^2}{\operatorname{ctg}B+\operatorname{ctg}C}+\dfrac{b^2}{\operatorname{ctg}C+\operatorname{ctg}A}+\dfrac{c^2}{\operatorname{ctg}A+\operatorname{ctg}B}\le 3pR$.

c) $\sum\dfrac{a^2\cos(B-C)}{\sin A}\le 4pR$.

1.8. $\dfrac{3}{64}\prod\left(1+\operatorname{ctg}^2\dfrac{A}{2}\right)+3\ge\sum\dfrac{\sec\dfrac{A}{2}+\operatorname{tg}\dfrac{A}{2}}{\sin A}$. *Kevin Soto Palacios, Huarmey, Peru*

1.9. a) $\sum m_a^2 r_a=p^2(R+4r)-r(4R+r)^2$.

b) $\dfrac{m_a^2}{r_br_c}+\dfrac{m_b^2}{r_cr_a}+\dfrac{m_c^2}{r_ar_b}=\dfrac{p^2(R+4r)-r(4R+r)^2}{rp^2}$.

1.10. $\dfrac{r_a}{\sin\dfrac{A}{2}}+\dfrac{r_b}{\sin\dfrac{B}{2}}+\dfrac{r_c}{\sin\dfrac{C}{2}}\ge 2\sqrt{3}s$. *Hoang Le Nhat Tung, Hanoi, Vietnam*

1.11. a) $\frac{1}{\sqrt{r_a}}+\frac{1}{\sqrt{r_b}}+\frac{1}{\sqrt{r_c}}+\frac{1}{\sqrt{r}}>\sqrt{\frac{2}{h_a}}+\sqrt{\frac{2}{h_b}}+\sqrt{\frac{2}{h_c}}$.

Nguyen Viet Hung, Hanoi, Vietnam

b) $\frac{1}{\sqrt{r_a}}+\frac{1}{\sqrt{r_b}}+\frac{1}{\sqrt{r_c}}\le\frac{1}{\sqrt{h_a}}+\frac{1}{\sqrt{h_b}}+\frac{1}{\sqrt{h_c}}$.

c) $\frac{1}{r_a^2}+\frac{1}{r_b^2}+\frac{1}{r_c^2}\ge\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}$.

d) $\frac{1}{r_a^3}+\frac{1}{r_b^3}+\frac{1}{r_c^3}\ge\frac{1}{h_a^3}+\frac{1}{h_b^3}+\frac{1}{h_c^3}$.

e) $\frac{1}{r_a^4}+\frac{1}{r_b^4}+\frac{1}{r_c^4}\ge\frac{1}{h_a^4}+\frac{1}{h_b^4}+\frac{1}{h_c^4}$.

f) $\frac{1}{r_a^n}+\frac{1}{r_b^n}+\frac{1}{r_c^n}\ge\frac{1}{h_a^n}+\frac{1}{h_b^n}+\frac{1}{h_c^n}$, $n\in\mathbb{N}$.

1.12. a) $a^3b^3+b^3c^3+c^3a^3=p^6+p^4\left(3r^2-12Rr\right)+3p^2r^4+r^3\left(4R+r\right)^3$.

b) $a^3b^3+b^3c^3+c^3a^3\ge16r^3\left(68R^3-69R^2r+30Rr^2-4r^3\right)$.

c) $a^3b^3+b^3c^3+c^3a^3\ge648R^3r^3$.

Seyran Ibrahimov, Maasilli, Azerbaidjan

1.13. a) Să se arate că pentru orice $x\in\mathbb{R}$ există un triunghi *ABC* având lungimile laturilor $c=AB=\sqrt{2x^2-2x+3}$, $b=AC=\sqrt{2x^2+2x+3}$, $a=BC=\sqrt{8x^2+10}$.

b) Aria triunghiului *ABC* este un număr irațional care nu depinde de *x*.

Iuliana Traşcă, Scorniceşti

c) Arătaţi că pentru orice $x\in\mathbb{R}$ există un triunghi *ABC* având lungimile laturilor $c=AB=\sqrt{\alpha x^2-\alpha x+\beta}$, $b=AC=\sqrt{\alpha x^2+\alpha x+\beta}$, $a=BC=\sqrt{4\alpha x^2+\gamma}$, unde $\alpha,\ \beta,\ \gamma>0$, $\alpha+\gamma=4\beta$ şi $\alpha<4\gamma$.

d) Aria triunghiului *ABC* nu depinde de *x*.

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1.14. $\sum\sin A\sin B\cos C\le\frac{9}{8}$, în triunghiul ascuţitunghic.

Neculai Stanciu, Buzău şi Titu Zvonaru, Comăneşti

1.15. a) $\sum\frac{a}{r_b^2+r_c^2}\le\frac{2R-r}{S}$.

Mehmet Şahin, Ankara, Turcia

b) $\sum\frac{a}{h_b^2+h_c^2}\le\frac{3R}{2S}$.

c) $\sum\frac{a}{h_bh_c}\le\sum\frac{a}{r_br_c}$.

1.16. a) $4\sum m_bm_c-4R\sum\frac{h_bh_c}{h_a}\le p^2+r\left(4R+r\right)$.

Bogdan Fuştei, România

b) $4\sum m_bm_c-4r\sum\frac{r_br_c}{r_a}\le p^2+5r\left(4R+r\right)$.

1.17. a) $\sum m_a\le6Rr\sum\frac{r_a}{bc}$.

b) $\frac{3}{2R}\le\sum\frac{r_a}{bc}\le\frac{3R}{8r^2}$.

c) $\sum m_a \le 3R^2 \sum \frac{h_a}{bc}$.

d) $\frac{3}{2R} \le \sum \frac{h_a}{bc} \le \frac{3}{4r}$.

e) $\sum \frac{h_a}{bc} \le \sum \frac{r_a}{bc}$.

f) $\frac{3}{2R} \le \sum \frac{h_a}{bc} \le \sum \frac{r_a}{bc} \le \frac{3R}{8r^2}$.

1.18. $4\left(l_a^2 + l_b^2 + l_c^2\right) \le (a+b+c)^2$. *Nguyen Viet Hung, Hanoi, Vietnam*

1.19. a) $\frac{AI \cdot h_a}{w_a(r_a+h_a)} + \frac{BI \cdot h_b}{w_b(r_b+h_b)} + \frac{CI \cdot h_c}{w_c(r_c+h_c)} = 1$. *Mustafa Tarek, Cairo, Egipt*

b) $\frac{w_a(r_a+h_a)}{AI \cdot h_a} + \frac{w_b(r_b+h_b)}{BI \cdot h_b} + \frac{w_c(r_c+h_c)}{CI \cdot h_c} \ge 9$. *Mustafa Tarek, Cairo, Egipt*

c) $\frac{w_a(r_a+h_a)}{AI \cdot h_a} + \frac{w_b(r_b+h_b)}{BI \cdot h_b} + \frac{w_c(r_c+h_c)}{CI \cdot h_c} = 1 + \frac{4R}{r}$.

d) $\frac{w_a(r_a+h_a)}{AI \cdot h_a} + \frac{w_b(r_b+h_b)}{BI \cdot h_b} + \frac{w_c(r_c+h_c)}{CI \cdot h_c} \le \frac{9R}{2r}$.

e) $9 \le \frac{w_a(r_a+h_a)}{AI \cdot h_a} + \frac{w_b(r_b+h_b)}{BI \cdot h_b} + \frac{w_c(r_c+h_c)}{CI \cdot h_c} \le \frac{9R}{2r}$.

1.20. a) $\sum \frac{1}{\tan\frac{A}{2} + \tan\frac{B}{2}} = \frac{p^2 + (4R+r)^2}{4Rp}$. b) $\sum \frac{1}{\cot\frac{A}{2} + \cot\frac{B}{2}} = \frac{p^2 + r^2 + 4Rr}{4Rp}$.

c) $\sum \left(\frac{1}{\tan\frac{A}{2} + \tan\frac{B}{2}} - \frac{1}{\cot\frac{A}{2} + \cot\frac{B}{2}} \right) \ge \sqrt{3}$. *Daniel Sitaru*

d) $\sum \frac{1}{1 - \tan\frac{A}{2}\tan\frac{B}{2}} = \frac{p^2 + r^2 + 4Rr}{4Rr}$. e) $\sum \frac{1}{\cot\frac{A}{2}\cot\frac{B}{2} - 1} = \frac{p^2 + r^2 - 8Rr}{4Rr}$.

f) $\sum \left(\frac{1}{1 - \tan\frac{A}{2}\tan\frac{B}{2}} + \frac{1}{1 - \cot\frac{A}{2}\cot\frac{B}{2}} \right) = 3$.

1.21. $(a+b-c)(b+c-a)(c+a-b) \le abc$. *Florentin Vişescu*

1.22. $l_a \le 2R\cos^2\frac{A}{2}\cos\frac{B-C}{2} \le m_a$.

1.23. a) $\frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c} \le 2\sqrt{3}\left(\frac{R}{2r}\right)^{\frac{3}{2}}$. *George Apostolopoulos, Grecia*

capitolul 2

Substituţiile lui Ravi

„Unica lecţie pe care o învăţăm la istorie este
că nimeni nu învaţă nicio lecţie din istorie.”
George Bernard Shaw

Propoziţie

Numerele $a,\ b,\ c$ sunt lungimile laturilor unui triunghi dacă şi numai dacă există $x,\ y,\ z \in \mathbb{R}_+^*$ astfel încât $a = y + z,\ b = z + x,\ c = x + y$. (Substituţiile lui Ravi)

Demonstraţie

Necesitate: Dacă a, b, c sunt lungimile laturilor unui triunghi, atunci:

$b + c - a > 0,\ c + a - b > 0,\ a + b - c > 0$ şi $x = \dfrac{b+c-a}{2},\ y = \dfrac{c+a-b}{2},\ z = \dfrac{a+b-c}{2}$

verifică $x,\ y,\ z \in \mathbb{R}_+^*$ astfel încât $a = y + z,\ b = z + x,\ c = x + y$.

Suficienţă: Dacă $a = y + z,\ b = z + x,\ c = x + y$, unde $x,\ y,\ z \in \mathbb{R}_+^*$, rezultă că $a,\ b,\ c > 0$ şi $a + b > c,\ b + c > a,\ c + a > b$, deci $a,\ b,\ c$ sunt lungimile laturilor unui triunghi.

În concluzie, dacă $a,\ b,\ c$ sunt lungimile laturilor unui triunghi, atunci pentru orice inegalitate de forma $F(a,\ b,\ c,\ p,\ R,\ r,\ S) \geq 0$ există inegalitatea duală $F'(x,\ y,\ z) \geq 0$, unde $x = p - a,\ y = p - b,\ z = p - c,\ R = \dfrac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}},\ r = \sqrt{\dfrac{xyz}{x+y+z}}$, $S = \sqrt{xyz(x+y+z)}$. Şi, reciproc, pentru orice inegalitate de forma $G(x,\ y,\ z) \geq 0$ există inegalitatea duală $G'(a,\ b,\ c,\ p, R,\ r,\ S) \geq 0$, unde $p,\ R,\ r,\ S$ reprezintă semiperimetrul, raza cercului circumscris, raza cercului înscris şi, respectiv, aria triunghiului cu laturile de lungimi $a,\ b,\ c$ şi $a = y + z,\ b = z + x,\ c = x + y$.

În continuare sunt propuse aplicaţii ale metodei duale.

Să se arate că în orice triunghi au loc inegalităţile:

2.1. $\prod\left(\sqrt{b} + \sqrt{c} - \sqrt{a}\right) \geq \sqrt{\prod(b + c - a)}$. *Lucian Stamate, Bucureşti*

2.2. a) $\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\ge 3\left(\sqrt[3]{\frac{4R}{r}}-1\right)$.

b) $\frac{r_a^2}{r_b^2}+\frac{r_b^2}{r_c^2}+\frac{r_c^2}{r_a^2}+\frac{8r}{R}\ge 7$. *RMM 8/2019, Rahim Shahbazov*

c) $\frac{r_a^2}{r_b^2}+\frac{r_b^2}{r_c^2}+\frac{r_c^2}{r_a^2}+\frac{2nr}{R}\ge n+3$, unde $n\le 4$.

d) $\frac{r_a^3}{r_b^3}+\frac{r_b^3}{r_c^3}+\frac{r_c^3}{r_a^3}+\frac{2nr}{R}\ge n+3$, unde $n\le 6$.

e) $\left(\frac{r_a}{r_b}\right)^4+\left(\frac{r_b}{r_c}\right)^4+\left(\frac{r_c}{r_a}\right)^4+\frac{2nr}{R}\ge n+3$, unde $n\le 8$.

f) $\left(\frac{r_a}{r_b}\right)^5+\left(\frac{r_b}{r_c}\right)^5+\left(\frac{r_c}{r_a}\right)^5+\frac{2nr}{R}\ge n+3$, unde $n\le 10$.

g) $\frac{r_a^3}{r_b^3}+\frac{r_b^3}{r_c^3}+\frac{r_c^3}{r_a^3}+\frac{54r}{4R+r}\ge 9$. *RMM 8/2019, Rahim Shahbazov*

h) $\frac{r_a^3}{r_b^3}+\frac{r_b^3}{r_c^3}+\frac{r_c^3}{r_a^3}+\frac{9nr}{4R+r}\ge n+3$, unde $n\le\frac{27}{4}$.

i) $\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\ge 3\sqrt[3]{\frac{2(4R+r)^2}{3p^2}-1}$.

j) $\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\ge\sqrt[3]{\frac{n(4R+r)^2}{p^2}+27-3n}$, unde $0\le n\le 27$.

k) $\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+\frac{4r}{R}\ge 5$. *RMM 8/2019, Rahim Shahbazov*

l) $\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}+\frac{2nr}{R}\ge n+3$, unde $n\le 2$.

m) $\frac{h_a}{h_b}+\frac{h_b}{h_c}+\frac{h_c}{h_a}\ge 3\left(\sqrt[3]{9-\frac{2r}{R}}-1\right)$.

n) $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge 3\left(\sqrt[3]{9-\frac{2r}{R}}-1\right)$.

o) $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\le\frac{4R^2+2r^2}{3Rr}$.

p) $\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\le\frac{4}{3}\left(\frac{R}{r}\right)^2-\frac{7}{3}$.

r) $5-\frac{4r}{R}\le\frac{r_a}{r_b}+\frac{r_b}{r_c}+\frac{r_c}{r_a}\le\frac{4}{3}\left(\frac{R}{r}\right)^2-\frac{7}{3}$.

s) $3+\frac{r(R-2r)}{8R^2}\le\frac{h_a}{h_b}+\frac{h_b}{h_c}+\frac{h_c}{h_a}\le\frac{1}{3}\left(\frac{R}{r}+1\right)^2$.

2.3. Fie $x, y, z > 0$. Arătaţi că:

a) $\dfrac{(x+y)^2(y+z)^2(z+x)^2}{(x+y+z)^3} \ge \dfrac{64}{27}xyz$. *Jalil Hajimir, Toronto, Canada*

b) $\left[\dfrac{3(x+y)(y+z)(z+x)}{8(x+y+z)}\right]^3 \ge (xyz)^2$.

2.4. a) $h_a + h_b + h_c \le r_a + r_b + r_c$.

b) $p\sqrt{3} \le r_a + r_b + r_c \le \dfrac{p^2}{3r}$.

GM 11/1971, Al. Popescu-Zorica, OIM 1971, SHL

2.5. $\dfrac{a+b+c}{\sqrt{a^2+b^2+c^2}} \le 2\dfrac{\sqrt{r_a^2+r_b^2+r_c^2}}{r_a+r_b+r_c-3r}$.

JBMO TST 2012, Turcia

2.6. $\dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b} + \dfrac{r}{R} \le 2$.

2.7. $\dfrac{1}{p(p-a)} + \dfrac{1}{p(p-b)} + \dfrac{1}{p(p-c)} \le \dfrac{ab+bc+ca}{4S^2}$. *100 Problems*

2.8. $\dfrac{1}{\sin\frac{A}{2}} + \dfrac{1}{\sin\frac{B}{2}} + \dfrac{1}{\sin\frac{C}{2}} \ge 4\sqrt{\dfrac{R}{r}}$.

2.9. a) $\dfrac{b+c-a}{\sqrt{a}} + \dfrac{c+a-b}{\sqrt{b}} + \dfrac{a+b-c}{\sqrt{c}} \ge \sqrt{3(a+b+c)}$.

b) $\dfrac{b+c-a}{\sqrt{a+c}} + \dfrac{c+a-b}{\sqrt{b+a}} + \dfrac{a+b-c}{\sqrt{c+b}} \ge \sqrt{\dfrac{3(a+b+c)}{2}}$.

c) $\dfrac{b+c-a}{\sqrt{a+2c}} + \dfrac{c+a-b}{\sqrt{b+2a}} + \dfrac{a+b-c}{\sqrt{c+2b}} \ge \sqrt{a+b+c}$.

RMT 4/2014, Andi Gabriel Brojbeanu, Târgovişte

d) $\dfrac{b+c-a}{\sqrt{a+nc}} + \dfrac{c+a-b}{\sqrt{b+na}} + \dfrac{a+b-c}{\sqrt{c+nb}} \ge \sqrt{\dfrac{3(a+b+c)}{n+1}}$, unde $0 \le n \le 2$.

Dezvoltare, Marin Chirciu

e) $\sqrt{\dfrac{a}{b+c-a}} + \sqrt{\dfrac{b}{c+a-b}} + \sqrt{\dfrac{c}{a+b-c}} \le 1 + \dfrac{R}{r}$. *V. Nicula, 2014*

capitolul 3

Inegalitatea lui Bergström; Cauchy–Buniakovski–Schwarz

„Cea mai distrugătoare explozie este explozia de entuziasm la un prost.”

Inegalitatea Cauchy–Buniakovski–Schwarz

Dacă $a_1, a_2, ..., a_n$ şi $b_1, b_2, ..., b_n$ sunt numere reale, atunci are loc inegalitatea:

$$\left(a_1b_1 + a_2b_2 + ... + a_nb_n\right)^2 \le \left(a_1^2 + a_2^2 + ... + a_n^2\right)\left(b_1^2 + b_2^2 + ... + b_n^2\right).$$

Egalitatea are loc dacă şi numai dacă $b_1 = \lambda a_1,\ b_2 = \lambda a_2,\ ...,\ b_n = \lambda a_n$, cu $\lambda \in \mathbb{R}$.

Corolar

i) Dacă a, b, x, y sunt numere reale şi $x, y > 0$, atunci:

$$\frac{a^2}{x} + \frac{b^2}{y} \ge \frac{(a+b)^2}{x+y}, \text{ cu egalitate dacă şi numai dacă } \frac{a}{x} = \frac{b}{y}.$$

ii) Dacă a, b, c, x, y, z sunt numere reale şi $x, y, z > 0$, atunci:

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \ge \frac{(a+b+c)^2}{x+y+z}, \text{ cu egalitate dacă şi numai dacă } \frac{a}{x} = \frac{b}{y} = \frac{c}{z}.$$

iii) Dacă $a_1, a_2, ..., a_n, x_1, x_2, ..., x_n$ sunt numere reale şi $x_1, x_2, ..., x_n > 0$, atunci:

$$\frac{a_1^2}{x_1} + \frac{a_1^2}{x_1} + ... + \frac{a_1^2}{x_1} \ge \frac{(a_1 + a_2 + ... + a_n)^2}{x_1 + x_2 + ... + x_n},$$

cu egalitate dacă şi numai dacă $\frac{a_1}{x_1} = \frac{a_2}{x_2} = ... = \frac{a_n}{x_n}$.

Corolarul este *inegalitatea lui Bergström.*

În continuare sunt propuse aplicaţii ce pot fi rezolvate folosind inegalităţile de mai sus.

3.1. a) $\sum h_a^m \cdot \sqrt{r_b + r_c} \le \sqrt{2\left(1+\frac{R}{r}\right)\left(h_a^{2m+1}+h_b^{2m+1}+h_c^{2m+1}\right)}$, unde $m \in \mathbb{R}$.

b) $\sum r_a^m \cdot \sqrt{a(b+c)} \le \sqrt{12R\left(r_a^{2m+1}+r_b^{2m+1}+r_c^{2m+1}\right)}$, unde $m \in \mathbb{R}$.

c) $\sum h_a^m \cdot \sqrt{h_b + h_c} \le \sqrt{\frac{p^2+r^2-2Rr}{2Rr} \cdot \sum h_a^{2m+1}}$, unde $m \in \mathbb{R}$.

d) $\sum h_a^m \cdot \sqrt{a(b+c)} \le \sqrt{\frac{p^2+r^2-2Rr}{r} \cdot \left(h_a^{2m+1}+h_b^{2m+1}+h_c^{2m+1}\right)}$, unde $m \in \mathbb{R}$.

e) $\sum h_a^m \cdot \sqrt{b+c} \le \sqrt{\frac{p^2+r^2+4Rr}{rp} \cdot \sum h_a^{2m+1}}$, unde $m \in \mathbb{R}$.

f) $\sum a^m \cdot \sqrt{b+c} \le \sqrt{\frac{p^2+r^2-2Rr}{2Rr} \cdot \sum a^{2m+1}}$, unde $m \in \mathbb{R}$.

g) $\sum\left(\cos\frac{B}{2}\cos\frac{C}{2}\right)^m \cdot \sqrt{\sin\frac{A}{2}} \le \sqrt{2 \cdot \sum\left(\cos\frac{B}{2}\cos\frac{C}{2}\right)^{2m+1}}$, unde $m \in \mathbb{R}$.

h) $\sum (r_b r_c)^m \cdot \sqrt{h_a(r_b+r_c)} \le \sqrt{6 \cdot \sum (r_b r_c)^{2m+1}}$, unde $m \in \mathbb{R}$.

3.2. a) $\sqrt{bc(p-a)} + \sqrt{ca(p-b)} + \sqrt{ab(p-c)} \le 3R\sqrt{p}$.

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b) $\sqrt{bc(p-a)} + \sqrt{ca(p-b)} + \sqrt{ab(p-c)} \le 2(R+r)\sqrt{p}$.

3.3. a) $a\sqrt{b} + b\sqrt{c} + c\sqrt{a} \le 3R\sqrt{2p}$. *Daniel Sitaru*

b) $a\sqrt{b} + b\sqrt{c} + c\sqrt{a} \le 2(R+r)\sqrt{2p}$.

c) $a\sqrt{b} + b\sqrt{c} + c\sqrt{a} \le 2(R+r)\sqrt{2p} \le 3R\sqrt{2p}$.

d) $a\sqrt{b+c} + b\sqrt{c+a} + c\sqrt{a+b} \le 4(R+r)\sqrt{p} \le 6R\sqrt{p}$.

e) $\sum a\sqrt{b+nc} \le 2(R+r)\sqrt{2(n+1)p} \le 3R\sqrt{2(n+1)p}$, unde $n \ge 0$.

3.4. a) $\sum \frac{\sin\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}} = 2\left(\frac{2R}{r}-1\right)$. b) $6 \le \sum \frac{1}{\sin\frac{A}{2}} \le 2\left(\frac{R}{r}+1\right)$.

c) $\sqrt{\frac{4R}{r}+10} \le \sum \sqrt{\frac{\sin\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}}} \le \sqrt{\frac{8R}{r}+2}$.

3.5. a) $\sum \dfrac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} = \dfrac{2(4R+r)}{p}$. b) $\dfrac{18r}{p} \le \sum \dfrac{1}{\cos\frac{A}{2}} \le \dfrac{2p}{3r}$.

c) $\dfrac{18r}{\sqrt{6S}} \le \sum \sqrt{\dfrac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}} \le \dfrac{9R}{\sqrt{6S}}$.

3.6. a) $\sum \dfrac{\operatorname{tg}\frac{A}{2}}{\operatorname{tg}\frac{B}{2}\operatorname{tg}\frac{C}{2}} = \dfrac{p}{r}\left[\left(\dfrac{4R+r}{p}\right)^2 - 2\right]$. b) $\prod \operatorname{ctg}\dfrac{A}{2} = \dfrac{p}{r}$.

c) $\dfrac{1}{r}\sqrt{3S} \le \sum \sqrt{\dfrac{\operatorname{tg}\frac{A}{2}}{\operatorname{tg}\frac{B}{2}\operatorname{tg}\frac{C}{2}}} \le \dfrac{1}{r}\sqrt{p(R+r)}$.

3.7. a) $\sum \dfrac{\operatorname{ctg}\frac{A}{2}}{\operatorname{ctg}\frac{B}{2}\operatorname{ctg}\frac{C}{2}} = \dfrac{p^2-2r^2-8Rr}{pr}$. b) $\prod \operatorname{tg}\dfrac{A}{2} = \dfrac{r}{p}$.

c) $3^{\frac{3}{4}} \le \sum \sqrt{\dfrac{\operatorname{ctg}\frac{A}{2}}{\operatorname{ctg}\frac{B}{2}\operatorname{ctg}\frac{C}{2}}} \le 3^{\frac{1}{4}} \cdot \sqrt{\dfrac{3R}{2r}}$.

3.8. a) $\dfrac{b+c}{a^2} + \dfrac{c+a}{b^2} + \dfrac{a+b}{c^2} = \dfrac{p^4 + p^2(2r^2-10Rr) + r^2(4R+r)(2R+r)}{8pR^2r^2}$.

b) $\dfrac{b+c}{a^5} + \dfrac{c+a}{b^5} + \dfrac{a+b}{c^5} \ge \dfrac{2}{3R^4}$.

D.M. Bătineţu-Giurgiu, Bucureşti şi Neculai Stanciu, Buzău

3.9. a) $\left(\dfrac{1}{m_a} + \dfrac{1}{m_b} + \dfrac{1}{m_c}\right)^2 \ge \dfrac{108}{5p^2-3r^2-12Rr}$.

b) $\dfrac{a^2+b^2+c^2}{a+b+c}\left(\dfrac{1}{m_a} + \dfrac{1}{m_b} + \dfrac{1}{m_c}\right) \ge 2\sqrt{3}$. *Nguyen Viet Hung, Hanoi, Vietnam*

c) $\left(\dfrac{1}{m_a+m_b} + \dfrac{1}{m_b+m_c} + \dfrac{1}{m_c+m_a}\right)^2 \ge \dfrac{36}{7p^2-5r^2-20Rr}$.

d) $\dfrac{a^2+b^2+c^2}{a+b+c}\left(\dfrac{1}{m_a+m_b}+\dfrac{1}{m_b+m_c}+\dfrac{1}{m_c+m_a}\right)\ge\sqrt{3}$.

e) $\left(\sum\dfrac{1}{m_a+\lambda m_b}\right)^2\ge\dfrac{108}{\left(5\lambda^2+11\lambda+5\right)p^2-\left(3\lambda^2+9\lambda+3\right)r^2-\left(12\lambda^2+36\lambda+12\right)Rr}$,

unde $\lambda\ge 0$.

f) $\dfrac{a^2+b^2+c^2}{a+b+c}\left(\dfrac{1}{m_a+\lambda m_b}+\dfrac{1}{m_b+\lambda m_c}+\dfrac{1}{m_c+\lambda m_a}\right)\ge\dfrac{2\sqrt{3}}{1+\lambda}$, unde $\lambda\ge 0$.

3.10. a) $\sum\sqrt{\cos A\sin B\sin C}\le\dfrac{3}{2}\sqrt{\dfrac{3}{2}}$, în triunghiul ascuţitunghic.

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b) $\sum\sqrt{\sin A\cos B\cos C}\le\dfrac{3}{2}\sqrt{\dfrac{\sqrt{3}}{2}}$.

3.11. a) $\dfrac{m_a}{m_b}+\dfrac{m_b}{m_c}+\dfrac{m_c}{m_a}\ge\dfrac{3\sum a^2}{2\sum a^2+\sum bc}+2$.

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b) $\dfrac{m_a}{m_b+m_c}+\dfrac{m_b}{m_c+m_a}+\dfrac{m_c}{m_a+m_b}\ge\dfrac{3\sum a^2}{4\sum a^2+2\sum bc}+1$.

c) $\dfrac{m_a}{m_b+\lambda m_c}+\dfrac{m_b}{m_c+\lambda m_a}+\dfrac{m_c}{m_a+\lambda m_b}\ge\dfrac{1}{\lambda+1}\left(\dfrac{3\sum a^2}{2\sum a^2+\sum bc}+2\right)$, unde $\lambda\ge 0$.

3.12. a) $\dfrac{r_a^3}{r_b}+\dfrac{r_b^3}{r_c}+\dfrac{r_c^3}{r_a}\ge\dfrac{(4R+r)^2(R-r)}{2R-r}$. *Adil Abdullayev, Baku, Azerbaidjan*

b) $\dfrac{r_a^3}{r_b+r_c}+\dfrac{r_b^3}{r_c+r_a}+\dfrac{r_c^3}{r_a+r_b}\ge\dfrac{(4R+r)^2(R-r)}{2(2R-r)}$.

c) $\dfrac{r_a^3}{r_b+\lambda r_c}+\dfrac{r_b^3}{r_c+\lambda r_a}+\dfrac{r_c^3}{r_a+\lambda r_b}\ge\dfrac{1}{\lambda+1}\cdot\dfrac{(4R+r)^2(R-r)}{2R-r}$, unde $\lambda\ge 0$.

d) $\dfrac{h_a^3}{h_b}+\dfrac{h_b^3}{h_c}+\dfrac{h_c^3}{h_a}\ge\left(\dfrac{2r}{R}\right)^2\left(9R^2-5Rr+r^2\right)$.

e) $\dfrac{h_a^3}{h_b+h_c}+\dfrac{h_b^3}{h_c+h_a}+\dfrac{h_c^3}{h_a+h_b}\ge\dfrac{1}{2}\left(\dfrac{2r}{R}\right)^2\left(9R^2-5Rr+r^2\right)$.

f) $\dfrac{h_a^3}{h_b+\lambda h_c}+\dfrac{h_b^3}{h_c+\lambda h_a}+\dfrac{h_c^3}{h_a+\lambda h_b}\ge\dfrac{1}{\lambda+1}\left(\dfrac{2r}{R}\right)^2\left(9R^2-5Rr+r^2\right)$, unde $\lambda\ge 0$.