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Daniel Sitaru

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**ALGEBRAIC PHENOMENON**

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# Chapter

# 1

## Famous Theorems

### CAUCHY-SCHWARZ's-Inequality

$$(ax + by)^2 \leq (a^2 + b^2)(x^2 + y^2); a, b, x, y \in \mathbb{R}$$

$$(ax + by + cz)^2 \leq (a^2 + b^2 + c^2)(x^2 + y^2 + z^2); a, b, c, x, y, z \in \mathbb{R}$$

$$\left( \sum_{i=1}^n a_i x_i \right)^2 \leq \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n x_i^2 \right); a_i, x_i \in \mathbb{R}; i \in \overline{1, n}$$

$$\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}; a, b \in \mathbb{R}; x, y \in (0, \infty)$$

$$\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \geq \frac{(a+b+c)^2}{x+y+z}; a, b, c \in \mathbb{R}; x, y, z \in (0, \infty)$$

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} \geq \frac{(a+b+c)^2}{ax+by+cz}; a, b, c, x, y, z \in (0, \infty)$$

$$\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \dots + \frac{a_n^2}{x_n} \geq \frac{(a_1 + a_2 + \dots + a_n)^2}{x_1 + x_2 + \dots + x_n}; a_i \in \mathbb{R}; x_i > 0; i \in \overline{1, n}$$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}; (\forall) a, b, c \in (0, \infty)$$

$$\frac{a}{b+nc} + \frac{b}{c+na} + \frac{c}{a+nb} \geq \frac{3}{n+1}; a, b, c \in (0, \infty); n \in \mathbb{N}^*$$

### MINKOWSKI's-Inequality

$$\sqrt{(x+a)^2 + (y+b)^2} \leq \sqrt{x^2 + y^2} + \sqrt{a^2 + b^2}$$

$$\sqrt{(x+y+z)^2 + (a+b+c)^2} \leq \sqrt{x^2 + a^2} + \sqrt{y^2 + b^2} + \sqrt{z^2 + c^2}$$

$$\sqrt{(x+a)^2 + (y+b)^2 + (z+c)^2} \leq \sqrt{x^2 + y^2 + z^2} + \sqrt{a^2 + b^2 + c^2}$$

$$\sqrt{(x_1+a_1)^2 + (x_2+a_2)^2 + \dots + (x_n+a_n)^2} \leq \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} + \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$$

$$x_i; a_i \in \mathbb{R}; i \in \overline{1, n}; n \in \mathbb{N}^*$$

$$\left( \sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}}$$

$$p > 1; x_i, y_i \in \mathbb{R}; i \in \overline{1, n}; n \in \mathbb{N}^*$$

# Chapter

# 2

## Questions

1. Find the minimum of the expression:

$$E = \frac{4ab - 11a^2 - 14b^2}{3(a^2 + b^2)}; a, b \in \mathbb{R}^*$$

2. Prove that if  $a, b, c \in (0, \infty)$  then:

$$\sum a^2 \left( \frac{1}{2016a^2 + 2015b^2} + \frac{1}{2016a^2 + 2015b^2} \right) > \frac{3}{4031}$$

3. Prove that if  $a, b, c \in (0, \infty)$ ;  $m \in \mathbb{R}^*$ ,  $a + b + c = \frac{2}{m^2}$  then:

$$\sqrt{m^2a + 1} + \sqrt{m^2b + 1} + \sqrt{m^2c + 1} \leq 4$$

4. Prove that if  $a, b, c \in \left(\frac{1}{k}, \infty\right)$ ;  $k > 0$ ,  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 2k$  then:

$$\sqrt{ka - 1} + \sqrt{kb - 1} + \sqrt{kc - 1} \geq \sqrt{a + b + c}$$

5. Prove that if  $x, y, z \geq 0$ ;  $t \in \mathbb{R}$ ;  $n \in \mathbb{N}$ ,  $x + y + z = 6t$  then:

$$x\sqrt{y+n} + y\sqrt{z+n} + z\sqrt{x+n} \leq \sqrt{3}(6t^2 + (3n+1)t)$$

6. Prove that if  $x, y, z \geq 0$  și  $x + y + z = 2$  then:

$$x\sqrt{y+1} + y\sqrt{z+1} + z\sqrt{x+1} \leq 2\sqrt{3}$$

7. Prove that if  $a, b, c, d \in (0, \infty)$ ;  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{d}$  then:

$$\frac{(a+b)(b+c)(c+a)}{(a-d)(b-d)(c-d)} \geq 27$$

8. Prove that if  $a, b, c \in (0, \infty)$  then:

$$\left(\frac{1}{a} + \frac{1}{b}\right) \left(\frac{1}{b} + \frac{1}{c}\right) \left(\frac{1}{c} + \frac{1}{a}\right) + \frac{1}{abc} \geq \frac{81}{(a+b)(b+c)(c+a) + abc}$$

# Chapter 3 Solutions

1. Find the minimum of the expression:

$$E = \frac{4ab - 11a^2 - 14b^2}{3(a^2 + b^2)}; a, b \in \mathbb{R}^*$$

Proof:

$$\begin{aligned} E &= \frac{4ab - 11a^2 - 14b^2}{3(a^2 + b^2)} = \frac{4a^2 + b^2 + 4ab - 15a^2 - 15b^2}{3(a^2 + b^2)} = \\ &= \frac{(2a+b)^2 - 15(a^2+b^2)}{3(a^2+b^2)} = \frac{(2a+b)^2}{3(a^2+b^2)} - 5 \geq -5, \quad \min E = -5 \text{ for } 2a + b = 0 \end{aligned}$$

2. Prove that if:  $a, b, c \in (0, \infty)$  then:

$$\sum a^2 \left( \frac{1}{2016a^2 + 2015b^2} + \frac{1}{2016a^2 + 2015b^2} \right) > \frac{3}{4031}$$

Proof:

$$\frac{a^2}{2016a^2 + 2015b^2} + \frac{b^2}{2016b^2 + 2015a^2} > \frac{(a+b)^2}{4031(a^2+b^2)} > \frac{1}{4031} \cdot \frac{(a+b)^2}{a^2+b^2} > \frac{1}{4031}$$

$$\text{because } \frac{(a+b)^2}{a^2+b^2} > 1; (a+b)^2 > a^2+b^2, a^2+2ab+b^2 > a^2+b^2; 2ab > 0$$

Analogous:

$$\frac{a^2}{2016a^2 + 2015c^2} + \frac{c^2}{2016c^2 + 2015a^2} > \frac{1}{4031}$$

$$\frac{b^2}{2016b^2 + 2015c^2} + \frac{c^2}{2016c^2 + 2015b^2} > \frac{1}{4031}$$

$$\sum a^2 \left( \frac{1}{2016a^2 + 2015b^2} + \frac{1}{2016a^2 + 2015b^2} \right) > \frac{3}{4031}$$

3. Prove that if:  $a, b, c \in (0, \infty); m \in \mathbb{R}^*, a + b + c = \frac{2}{m^2}$  then:

$$\sqrt{m^2a + 1} + \sqrt{m^2b + 1} + \sqrt{m^2c + 1} \leq 4$$

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